

Pairing Effects in the Edge of Paired Quantum Hall States

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We study pairing effects in the edge states of paired fractional quantum Hall states by using persistent edge currents as a probe. We give the grand partition functions for edge excitations of paired states (Haldane-Rezayi, Pfaffian, 331) coupling to an Aharonov-Bohm flux and derive the exact formulas of the persistent edge current. We show that the currents are flux periodic with the unit flux $\phi_0 = hc/e$. At low temperatures, they exhibit anomalous oscillations in their flux dependence. The shapes of the functions depend on the bulk topological order. They converge to the sawtooth function with period $\phi_0/2$ at zero temperature, which indicates pair condensation. This phenomenon provides an interesting bridge between superconductivity in 2+1 dimensions and superconductivity in 1+1 dimensions. We propose experiments of measuring the persistent current at even denominator plateau in single or double layer systems to test our predictions.

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One of the surprising aspects of the fractional quantum Hall effect is that the edge state forms a new kind of state of matter beyond Fermi liquid, called the chiral Tomonaga-Luttinger liquid [1]. Some experiments have already demonstrated the characteristic behavior of chiral Tomonaga-Luttinger liquid [2,3]. Recently the Aharonov-Bohm effect (AB effect) in such systems were studied by Geller et al [4,5] and Chamon et al [6]. Especially, the former authors predict new edge-current oscillations in the persistent current at the edge of the $\nu = \frac{1}{q}$ Laughlin state, which has no amplitude reduction from disorder and thus results in a universal non-Fermi-liquid temperature dependence [5]. Also the persistent current for the annulus Laughlin state was recently investigated by Kettmann [7]. These studies show the current is periodic with a unit flux quanta $\phi_0 = hc/e$ in agreement with the theorem of Byers and Young [8].

Motivated by these recent studies, we investigate the persistent currents in paired fractional Hall states in this paper. The states we will consider are the 331 state [9], the Haldane-Rezayi state [10] and the Pfaffian state [11]. They are quantum Hall analogs of the BCS superconductor. The pairing symmetry is p-wave with $S_z = 1, 0$ for the Pfaffian and the 331 states respectively, and d-wave for the Haldane-Rezayi states. The Haldane-Rezayi state and the Pfaffian state are new kinds of quantum Hall state which recently attracted considerable attentions because they are supposed to exhibit some novel features beyond ordinary quantum Hall state in their topological ordering, such as nonabelian statistics and specific degeneracy on a surface with nontrivial topology. On the other hand, the 331 state is a part of generalized hierarchy [12], but can be interpreted as a paired state. These states are proposed as a candidate for the $\nu = \frac{5}{2}$ plateau in single

layer systems [13] and the $\nu = \frac{1}{2}$ plateau in double layer systems [14]. Recent numerical study by Morf suggests that the $\nu = \frac{5}{2}$ state is Pfaffian-like [15].

As in the Laughlin state, these states are incompressible and have edge excitations. The edge states have a richer content than chiral Tomonaga-Luttinger liquid due to their internal degree of freedom [16–20]. Since one cannot see into the bulk state directly, it is important to know how the edge state reflects the bulk topological order. Thus we will compare the persistent edge currents for paired states to see how the pairing of the electrons in the bulk affects the properties of the edge states. Since bulk states of paired states are supposed to be in a superconducting phase, one naively imagine that the persistent edge current should have a period $\phi_0/2 = hc/2e$. However the fact that there is no spontaneous symmetry breaking of continuous symmetry in $1 + 1$ dimensional quantum field theory prevent this naive guess to be right. We will examine the property of pairing in the edge state by using the edge conformal field theory.

Let us first recall edge excitations on a single edge of the $\nu = \frac{1}{q}$ Laughlin state. The edge state of the Laughlin state is a chiral Tomonaga-Luttinger liquid which is described by a chiral boson φ . Its Hilbert space is described by the $U(1)$ Kac-Moody algebra generated by $j = \frac{1}{\sqrt{q}}\partial\varphi$ and the zero modes which correspond to quasiparticles. The Hamiltonian is given by $H = \frac{1}{2}\sum_{n\in\mathbb{Z}}j_nj_{-n} - \frac{c}{24}$ where we include a Casimir factor with $c = 1$. The complete description of the edge excitations can be given by a Virasoro character which corresponds to the partition function of the grand ensemble of quasiparticles. For the edge state of the Laughlin state, the Virasoro character is given by

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$$\chi_q(\tau) = \frac{1}{\eta} \sum_{n \in \mathbf{Z}} \exp\left(\frac{\pi i}{q} \tau n^2\right) \quad (1)$$

where η is the Dedekind function $\eta(\tau) = x^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - x^n)$, $x = e^{2\pi i \tau}$ and τ is a modular parameter. Let us modify (1) to get the grand partition function for the edge state coupling to an AB flux with finite size effects. Let L be the circumference of the edge state. The finite size L induces a temperature scale $T_0 = \frac{\hbar v}{k_B L}$ where v is the Fermi velocity of the edge modes determined by the confining potential. For example, a Fermi velocity of 10^6 cm/s and circumference of 1μ m yields $T_0 \sim 60$ mK. The coupling of the edge state to an AB flux Φ induces a twisted boundary condition on the chiral boson φ . Accordingly the grand partition function becomes [21,5]

$$\begin{aligned} Z_{\text{Laugh}}(\tau, \phi) &= \chi_q(\tau, \phi) = \frac{1}{\eta} \sum_{n \in \mathbf{Z}} \exp\left(\frac{\pi i}{q} \tau (n - \phi)^2\right) \\ &= \frac{\sqrt{q}}{\eta(t)} \theta_3(\phi|qt) \end{aligned} \quad (2)$$

where $\tau = i\frac{T_0}{T}$, $t = i\frac{T}{T_0}$, $\phi = \Phi/\phi_0$ with $\phi_0 = hc/e$ the unit flux quantum and θ_3 is the third Jacobi θ function. From the grand partition function Eq.(2), one can deduce the exact formula of the persistent current for a chiral Tomonaga-Luttinger liquid. In general, the persistent current I is defined by the following formula:

$$I \equiv \frac{T}{\phi_0} \frac{\partial \ln Z(\tau, \phi)}{\partial \phi}. \quad (3)$$

Then the persistent current in the Laughlin state is calculated to be

$$I_{\text{Laugh}} = \frac{2\pi T}{\phi_0} \sum_{n=0}^{\infty} (-1)^n \frac{\sin(2\pi n \phi)}{\sinh(nq\pi T/T_0)} \quad (4)$$

which is the formula obtained [5]. As the temperature is lowered, the shape as a function of ϕ changes from a sinusoidal to the sawtooth function,

$$I_{\text{Laugh}} \rightarrow \frac{T_0}{q\phi_0} \sum \frac{(-1)^m}{m} \sin 2\pi m \phi \quad (5)$$

$$= -\nu \frac{ev}{L} (\phi - r), \quad (6)$$

for $-\frac{1}{2} + r < \phi < \frac{1}{2} + r$, $r \in \mathbf{Z}$. The periodicity of I_{Laugh} in Φ is ϕ_0 , which agrees with the general theorem of Byers and Yang [8]. This is due to the presence of the quasiparticle with a fractional charge as argued in Refs. [5,7]. Since there is no backscattering from impurities in the chiral Luttinger liquid, the current has no reduction from impurities and therefore shows non-Fermi liquid dependence on the temperature.

Let us now consider paired quantum Hall states. We first deduce the grand partition function for the edge excitations of a single edge of the paired states coupling to an AB flux with the finite size effect.

In general, paired states have an extra internal degree of freedom other than the charge. In the bulk conformal field theory description, paired states have some kinds of fermion ψ for an internal degree of freedom and a chiral boson for the charge degree of freedom. The operator for the particles of the system is of the form $\psi e^{i\sqrt{q}\varphi}$ with q is even for the system of electrons. The edge excitations accordingly have a neutral sector corresponding to ψ .

Consider the Pfaffian state. The additional sector of edge excitations of this state is described by Majorana-Weyl fermion. Accordingly, it is generated by the $c = 1/2$ Virasoro algebra. Since there are degenerate states in its spectrum, these modes contribute to the partition function through the following Virasoro characters for Majorana-Weyl fermion:

$$\chi_1^{\text{MW}}(\tau) = \frac{1}{2} x^{-\frac{1}{48}} \left(\prod_0^{\infty} (1 + x^{n+\frac{1}{2}}) + \prod_0^{\infty} (1 - x^{n+\frac{1}{2}}) \right), \quad (7)$$

$$\chi_{\psi}^{\text{MW}}(\tau) = \frac{1}{2} x^{-\frac{1}{48}} \left(\prod_0^{\infty} (1 + x^{n+\frac{1}{2}}) - \prod_0^{\infty} (1 - x^{n+\frac{1}{2}}) \right), \quad (8)$$

$$\chi_{\sigma}^{\text{MW}}(\tau) = x^{\frac{1}{24}} \prod_1^{\infty} (1 + x^n). \quad (9)$$

χ_1^{MW} and χ_{ψ}^{MW} correspond to the untwisted sector of Majorana-Weyl fermion and $\chi_{\sigma}^{\text{MW}}$ to the twisted sector. The couplings of these sectors to the charge are determined by the condition of no monodromy with $\psi e^{i\sqrt{q}\varphi}$, which requires the untwisted sector couple to \mathbf{Z}/q charge modes and twisted sector to $(\mathbf{Z} + 1/2)/q$ charge modes. Therefore the couplings are given by following products of characters

$$\chi_1^{\text{MW}} \chi_q, \quad \chi_{\psi}^{\text{MW}} \chi_q, \quad \chi_{\sigma}^{\text{MW}} \chi_q^{(1/2)}, \quad (10)$$

where we have introduced a character

$$\chi_q^{(1/2)}(\tau) = \frac{1}{\eta} \sum_{n \in \mathbf{Z} + 1/2} \exp\left(\frac{\pi i}{q} \tau n^2\right). \quad (11)$$

Now the character of edge excitations of the Pfaffian state becomes

$$\chi_{\text{Pf}} = \chi_1^{\text{MW}} \chi_q + \chi_{\psi}^{\text{MW}} \chi_q + \chi_{\sigma}^{\text{MW}} \chi_q^{(1/2)}. \quad (12)$$

To deduce the grand partition functions with an AB flux, we note that the AB flux will couple only to the charged sector of the edge states and change the boundary condition of the chiral boson φ as in the Laughlin state, which changes the grand partition function of chiral boson to $\chi_q(\tau, \phi)$ of Eq.(2). Also $\chi_q^{(1/2)}(\tau)$ changes to

$$\begin{aligned} \chi_q^{(1/2)}(\tau, \phi) &= \frac{1}{\eta} \sum_{n \in \mathbf{Z} + 1/2} \exp\left(\frac{\pi i}{q} \tau (n - \phi)^2\right) \\ &= \frac{\sqrt{q}}{\eta(t)} \theta_4(\phi|qt), \end{aligned} \quad (13)$$

where $t = i\frac{T}{T_0}$. Then the grand partition functions with an AB flux can be written in terms of Jacobi θ functions as

$$Z_{\text{Pf}}(t, \phi) = \frac{\sqrt{q}}{\eta^{3/2}} \left[\sqrt{\theta_3(0|t)}\theta_3(\phi|qt) + \sqrt{\frac{\theta_2(0|t)}{2}}\theta_4(\phi|qt) \right] \quad (14)$$

Similarly the Haldane-Rezayi state and the 331 state have the internal degree of freedom described by the $c = -2$ scalar fermion and the $c = 1$ Dirac fermion respectively [18]. The grand partition functions become

$$Z_{\text{HR}}(t, \phi) = \frac{\sqrt{q}}{\eta^2} [\theta_4(0|t)\theta_3(\phi|qt) + \theta_3(0|t)\theta_4(\phi|qt)], \quad (15)$$

$$Z_{331}(t, \phi) = \frac{\sqrt{q}}{\eta^2} [\theta_3(0|t)\theta_3(\phi|qt) + \theta_4(0|t)\theta_4(\phi|qt)]. \quad (16)$$

Here the zero modes of fermions are included. We see that $Z_{\text{HR}}(\tau, \phi \pm 1/2) = Z_{331}(\tau, \phi)$. This implies that the 331 state continuously evolve into the Haldane-Rezayi state by activating an AB flux by half unit flux quantum.

Putting these formulas into Eq.(3), we get the exact formulas for the persistent current for the Pfaffian, Haldane-Rezayi, 331 states as

$$I_{\text{Pf}}(t, \phi) = \frac{T}{\phi_0} \frac{\sqrt{\theta_3(0|t)}\theta'_3(\phi|qt) + \sqrt{\frac{\theta_4(0|t)}{2}}\theta'_4(\phi|qt)}{\sqrt{\theta_3(0|t)}\theta_3(\phi|qt) + \sqrt{\frac{\theta_4(0|t)}{2}}\theta_4(\phi|qt)}. \quad (17)$$

$$I_{\text{HR}}(t, \phi) = \frac{T}{\phi_0} \frac{\theta_4(0|t)\theta'_3(\phi|qt) + \theta_3(0|t)\theta'_4(\phi|qt)}{\theta_4(0|t)\theta_3(\phi|qt) + \theta_3(0|t)\theta_4(\phi|qt)}, \quad (18)$$

$$I_{331}(t, \phi) = \frac{T}{\phi_0} \frac{\theta_3(0|t)\theta'_3(\phi|qt) + \theta_4(0|t)\theta'_4(\phi|qt)}{\theta_3(0|t)\theta_3(\phi|qt) + \theta_4(0|t)\theta_4(\phi|qt)}. \quad (19)$$

Here $\theta'(\phi|\tau)$ is the differential of $\theta(\phi|\tau)$ with respect to ϕ . We see $I_{\text{HR}}(t, \phi \pm 1/2) = I_{331}(t, \phi)$.

The analytic properties of these currents follows from the properties of θ functions. At low temperature $T \ll T_0$, the contribution from θ_3 which can be considered as from the chiral Tomonaga-Luttinger liquid is dominant in I around $\phi \sim 0$ and its periodic points. On the other hand, around $\phi \sim 0.5$, θ_4 which is from the pairing structure is dominant. This is because θ_3 (θ_4) is localized around points $\phi \in \mathbf{Z}$ ($\phi \in \mathbf{Z} + 1/2$) as the temperature is lowered. Therefore the currents converge at zero temperature to

$$I \rightarrow -\nu \frac{ev}{k_B L} \left(\phi - \frac{1}{2}r \right), \quad (20)$$

for $-\frac{1}{4} + \frac{1}{2}r < \phi < \frac{1}{4} + \frac{1}{2}r$, $r \in \mathbf{Z}$. At high temperatures, θ_3 and θ_4 are not localized and thereby their contributions are tamed. Thus the currents are suppressed compared to the current for the Laughlin state.

We especially take the simplest and experimentally important $\nu = 1/2$ case to show some plots. Similar behavior holds for arbitrary q . Fig.1 shows the flux oscillations

of persistent currents I_{Pf} for the $\nu = 1/2$ Pfaffian state at $T/T_0 = 0.6, 0.5, 0.45, 0.37, 0.3$. The current is periodic with period ϕ_0 , but the flux dependence exhibits an anomalous behavior in these cases. As the temperature is lowered, extra zero points appear and the shape as a function of ϕ is no longer a sinusoidal function. These extra zero points approach to the points $\pm\phi_0/2$ as the temperature is lowered. These points equal to $\pm\phi_0/2$ only at zero temperature and the shape as a function of ϕ converges to the sawtooth function in (20) which have a period $\phi_0/2$, which means that a pairing condensation occurs.

The currents for the Haldane-Rezayi state and the 331 state exhibit similar behaviors.

The currents at $T/T_0 = 0.45$ are compared in Fig.2. It shows that the shape of the oscillations is different for each state at finite temperatures and therefore depend on the bulk topological order. We also see that the currents are suppressed compared to the Laughlin state.

Fig.3 shows the temperature dependence of the persistent currents near the origin, $\phi = -0.05$. The currents decay exponentially as the temperature is highered. The magnitude of the currents for paired states decays faster than the one for the Laughlin state. The temperature dependence is also different among paired states.

Anomalous oscillations of the persistent currents are explained from the BCS pairing of electrons in the bulk and edge of paired states. Naively, the edge persistent currents which we have calculated should have a period $\phi_0/2$ since the bulk states of paired states are in a BCS superconducting phase and the order parameter has a charge $2e$. However as there is no spontaneous symmetry breaking of continuous symmetry in 1+1 dimensions, the edge states can not be a BCS condensate except at zero temperature. Then, from the behavior of the currents we have found, we see that as we lower the temperature, the edge states become closer to a BCS condensate, but the phase transition of the edge states does not occur at finite temperatures. Thus the BCS pairing structure in the edge states becomes stronger at lower temperatures, but the condensation only occurs at zero temperature.

This phenomenon may be seen as an interesting bridge between superconductivity in 2+1 dimensions and superconductivity 1+1 dimensions.

The suppression of the persistent current means that the increase of the viscosity of paired quantum Hall fluids from the hydrodynamical point of view which is often used to describe the edge excitations for the fractional quantum Hall droplet [1]. It implies that the presence of strong pairing of electrons increases the viscosity of the fluids beyond the Laughlin state.

Thus we see that the predicted behaviors of persistent edge currents can be used as a method to distinguish the bulk topological order. Especially the flux dependence can reveal the pairing of electrons in the bulk of fractional quantum Hall states.

Experimentally, the magnitude of the persistent currents is preferably measurable at low temperatures and small samples. As we have predicted, experiments at the even-denominator plateau, or in the double layers may detect the anomalous oscillations of the persistent current.

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$\nu = 1/2$ Pfaffian State

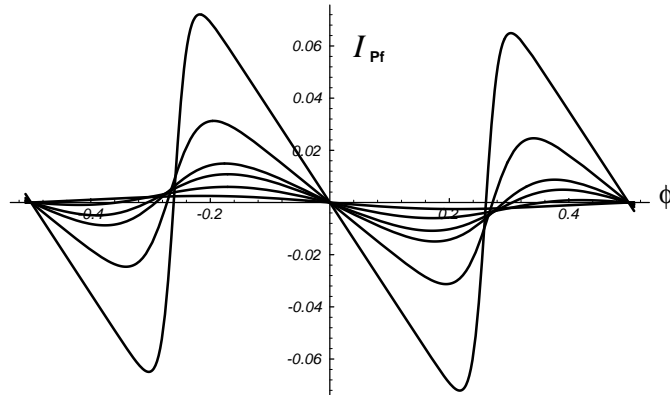


FIG. 1. The flux dependence of the persistent currents for the $\nu = 1/2$ Pfaffian state at temperatures $T/T_0 = 0.6, 0.5, 0.45, 0.45, 0.37, 0.3$. The currents are measured in the unit $\nu e v/2(k_B L)$

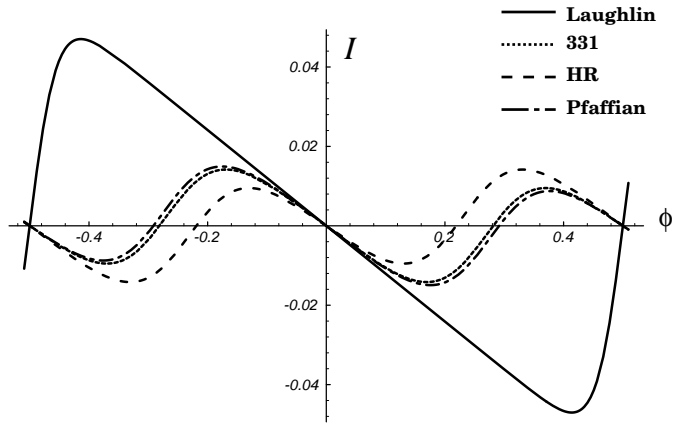


FIG. 2. The oscillations of persistent currents at $T/T_0 = 0.45$.

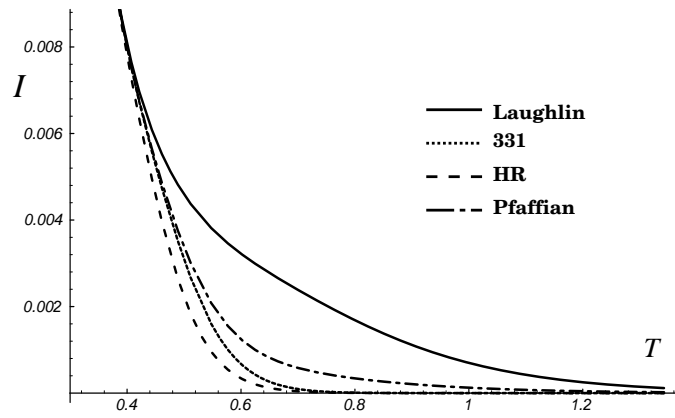


FIG. 3. Temperature dependence of persistent currents at $\phi = -0.05$.